

Spectrum of γ -Fluids: A Statistical Derivation

J. A. E. Carrillo,¹ J. A. S. Lima,^{2,3} and A. Maia, Jr.⁴

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The spectrum of massless bosonic and fermionic fluids satisfying the equation of state $p = (\gamma - 1)\rho$ is derived using elementary statistical methods. As a limiting case, the Lorentz-invariant spectrum of the vacuum ($\gamma = 0$, $p = -\rho$) is deduced. These results are in agreement with our earlier derivation for bosons using thermodynamics and semiclassical considerations.

1. INTRODUCTION

The class of γ -fluids comprises the simplest kind of relativistic perfect simple fluids used in the framework of general relativity and cosmology. Such a class is usually defined in terms of the so-called “ γ -law” equation of state

$$P = (\gamma - 1)\rho \quad (1)$$

where $\gamma \in [0, 2]$. Some special types of media described by the above relation are (i) the vacuum ($p = -\rho$, $\gamma = 0$), (ii) a randomly oriented distribution of infinitely thin, straight strings averaged over all directions ($p = -\frac{1}{3}\rho$, $\gamma = 2/3$), (iii) blackbody radiation ($p = \frac{1}{3}\rho$, $\gamma = 4/3$), and (iv) stiff matter ($p = \rho$, $\gamma = 2$). In a series of recent papers (Lima and Santos, 1995; Lima and Maia, 1995a,b), some general properties of this monoparametric family of fluids have been discussed based on thermodynamic and semiclassical considerations. In particular, we have stressed the unusual ther-

¹Instituto de Física “Gleb Wataghin”, UNICAMP, 13.081-970, Campinas, SP, Brazil; e-mail: espichan@ifi.unicamp.br.

²Department of Physics, Brown University, Providence, Rhode Island 02912; e-mail: limajas@het.brown.edu.

³Departamento de Física, UFRN, CP 1641, 59072-970, Natal, Rio Grande do Norte, Brazil.

⁴Instituto de Matemática, UNICAMP, 13.081-970, Campinas, S.P., Brazil; e-mail: maia@ime.unicamp.br.

modynamic behavior arising when the γ parameter is smaller than one. In this case, unlike the subset with positive pressure, the temperature increases in the course of an adiabatic expansion. In the vacuum case, for instance, it was shown that the temperature scales as $T \sim V$, where V is the volume. Further, by assuming that such fluids may be regarded as a kind of generalized radiation, the general Planck form of the spectrum has been obtained, which includes the vacuum spectrum as a particular case (Lima and Maia, 1995b). In our opinion, the special attention given to this class of fluids has a very simple physical motivation. Physicists have no intuitive picture of the relativistic quantum vacuum, which remains one of the most unknown physical systems. A possible way to overcome such a difficulty is using a γ -fluid. In principle, by establishing the physical properties for a generic value of γ , one can obtain the Lorentz-invariant vacuum properties taking the limit $\gamma = 0$. Such a possibility may be important even in the cosmological domain, where the vacuum physics is closely related to the cosmological constant problem (Weinberg, 1989). In this connection, it should be recalled that in some stages, the scalar fields driving inflation can also be thought of as a kind of γ -fluid, regardless of the details of its potential. This happens, for instance, during the coherent field oscillation phase of the inflaton field at the end of inflation (Kolb and Turner, 1990).

In this context, it seems interesting to extend the classical thermodynamic approach developed in the above papers, making the necessary connection with the microphysics underlying such systems. In the present article, our main goal is to show how the Planckian-type distribution for a γ -fluid, which has been discussed in the framework of the old quantum theory of radiation, can be reproduced in the domain of statistical mechanics. This allows us to extend the theory for fermions as well. Of course, the third and last step would be to derive the spectrum from a more basic theory such as quantum field theory.

2. THE SPECTRUM OF γ -FLUIDS

Now consider the canonical procedure to compute the pressure p and the energy density ρ in elementary statistical mechanics. As usual, these quantities are defined by

$$p = kT \left(\frac{\partial \ln Q}{\partial V} \right)_T \quad (2)$$

and

$$\rho = \frac{kT^2}{V} \left(\frac{\partial \ln Q}{\partial T} \right)_V \quad (3)$$

where $\ln Q$ is the grand-canonical thermodynamic potential, which corresponds to a quantum fluid in contact with a thermal reservoir at temperature T . Since we are assuming that the vacuum state behaves like a kind of radiation which differs from blackbody radiation only due to the equation of state, we take the chemical potential of any γ -fluid to be identically zero. In this case, by considering a continuous spectrum, we have the well-known formula (Itzykson and Zuber, 1980)

$$\ln Q = -V \int_0^\infty \ln \left[1 \mp \exp\left(-\frac{\hbar\omega}{kT}\right) \right] f(\omega) d\omega \tag{4}$$

where the upper and lower signs inside the brackets correspond to bosons and fermions, respectively.

Our aim now is to find the unknown function $f(\omega)$, which is the number of states per unit energy.

From equations (1)–(4) we get easily

$$-kT \int_0^\infty \ln \left[1 \mp \exp\left(-\frac{\hbar\omega}{kT}\right) \right] f(\omega) d\omega = (\gamma - 1)\hbar \int_0^\infty \frac{\omega f(\omega)}{\exp(\hbar\omega/kT) \mp 1} d\omega \tag{5}$$

Of course, the above equation points to a singularity at $\gamma = 1$. This rather pathological case (“dust”), describing a zero-pressure fluid, will not be considered here. A partial integration on the left-hand side of (5) furnishes

$$-kT \ln \left[1 \mp \exp\left(-\frac{\hbar\omega}{kT}\right) \right] F(\omega) \Big|_0^\infty + \hbar \int_0^\infty \frac{F(\omega)}{\exp(\hbar\omega/kT) \mp 1} d\omega \tag{6}$$

where $F(\omega)$ is a primitive of $f(\omega)$:

$$F'(\omega) = f(\omega) \tag{7}$$

Let us now suppose, for a moment, that the first term in (6), which corresponds to a boundary term, vanishes. In what follows, it will become clear under which conditions the function $f(\omega)$ will fulfill such a constraint. Bearing this in mind, we may write from (5) and (6)

$$\int_0^\infty \frac{F(\omega)}{\exp(\hbar\omega/kT) \mp 1} d\omega = (\gamma - 1) \int_0^\infty \frac{\omega f(\omega)}{\exp(\hbar\omega/kT) \mp 1} d\omega \tag{8}$$

The correctness of the above equation will be guaranteed if the functions $f(\omega)$ and $F(\omega)$ obeying (7) satisfy the following relation:

$$F(\omega) = (\gamma - 1)\omega f(\omega) \tag{9}$$

In principle, we cannot guarantee that equation (9) will furnish all physically meaningful solutions of equations (7) and (8). Our confidence that it is the physical solution is supported by our equivalent earlier result using only thermodynamic and semiclassical considerations (Lima and Maia, 1995b). In addition, it is easy to see that equation (9) is independent of the statistics of the γ -fluid particles.

From equations (7) and (9) one obtains the differential equation for $f(\omega)$,

$$\frac{f'(\omega)}{f(\omega)} = \left(\frac{2 - \gamma}{\gamma - 1} \right) \frac{1}{\omega} \quad (10)$$

where the prime denotes derivation with respect to ω . The solution of above equation is straightforward,

$$f(\omega) = A\omega^{(2-\gamma)/(\gamma-1)} \quad (11)$$

where A is a γ -dependent integration constant. Now, inserting the above equation into (4) and using (3), we obtain

$$\rho(T) = \int_0^\infty \frac{A\omega^{1/(\gamma-1)}}{\exp(\hbar\omega/kT) \mp 1} d\omega \quad (12)$$

Therefore, the spectrum of a γ -fluid reads

$$\rho(\omega, T) = \frac{A\omega^{1/(\gamma-1)}}{\exp(\hbar\omega/kT) \mp 1} \quad (13)$$

For the case of bosons, equations (12) and (13) above are, respectively, (39) and (53) presented by Lima and Maia (1995b). As expected, by introducing a new variable $x = \hbar\omega/kT$, one obtains from (12) the generalized Stefan–Boltzmann law (Lima and Santos, 1995)

$$\rho(T) = \eta T^{\gamma/(\gamma-1)} \quad (14)$$

where the constant η depends on the γ parameter as well as on the bosonic (or fermionic) spin degrees of freedom of each field. Note also that the above expression for $\rho(T)$ does not mean that the energy density is always finite for any value of γ . In particular, for the vacuum case ($\gamma = 0$), ρ effectively does not depend on the temperature, but the constant η is infinite, as it should be from quantum field theory.

3. THE VACUUM-INFRARED DIVERGENCE

Naturally, the validity of equations (12)–(14) is crucially dependent on our earlier hypotheses concerning the boundary term in equation (6). In order

to clarify this point we will compute explicitly such a term. From (9) and (11) it follows that

$$F(\omega) = A(\gamma - 1)\omega^{1/(\gamma-1)} \tag{15}$$

Now, inserting the function $F(\omega)$ into (6), we see that for γ ranging over the interval $1 < \gamma \leq 2$, the boundary term vanishes in accordance with our earlier conjecture. In particular, this means that the above derivation works well in the case of photons ($\gamma = 4/3$). However, we find a divergence in the limit $\omega \rightarrow 0$ when $0 \leq \gamma < 1$. In the vacuum case, for instance, equation (13) reduces to

$$\rho_{\text{vac}}(\omega, T) = \frac{A\hbar\omega^{-1}}{\exp(\hbar\omega/kT) \mp 1} \tag{16}$$

Thus, even though the vacuum energy density does not depend on the temperature [see (15)], it also exhibits the same kind of divergence. Thus the spectrum for negative pressures ($0 \leq \gamma < 1$) demands closer attention due to the inevitable existence of an infrared divergence.

To avoid the infrared catastrophe we proceed in analogy with the Casimir effect, in which the divergent energy density has been regularized by an ultraviolet exponential cutoff $e^{-\alpha\omega}$ with $\alpha > 0$ (Plunien *et al.*, 1986; Ruggiero and Zimmerman, 1977). In this way, we use an infrared exponential cutoff $e^{-\alpha/\omega}$, $\alpha > 0$. By introducing the regularized function

$$F_\alpha(\omega) = F(\omega)e^{-\alpha/\omega} \tag{17}$$

it is straightforward to check that $F_\alpha(\omega)$ makes the boundary term in (6) vanish.

Now, returning to equation (7), we may define its regularized counterpart

$$f_\alpha(\omega) = F_\alpha'(\omega) \tag{18}$$

and from (15), (17), and (18) we readily obtain the regularized density of states function

$$f_\alpha(\omega) = A\left(1 + \frac{(\gamma - 1)\alpha}{\omega}\right)\omega^{(2-\gamma)/(\gamma-1)} \exp\left(-\frac{\alpha}{\omega}\right) \tag{19}$$

which, as should be expected, reduces to $f(\omega)$ in the limit $\alpha \rightarrow 0$. From (5) and (6) the regularized equation of state reads

$$P_\alpha = (\gamma - 1)\rho_\alpha \tag{20}$$

where

$$P_\alpha = \int_0^\infty \frac{F_\alpha(\omega)}{\exp(\hbar\omega/kT) \mp 1} d\omega \tag{21}$$

and

$$\rho_\alpha = \int_0^\infty \frac{\omega f_\alpha(\omega)}{\exp(\hbar\omega/kT) \mp 1} d\omega \quad (22)$$

It should be noticed that the above regularized integrals are finite for all cases $0 \leq \gamma \leq 2$, $\gamma \neq 1$. However, if $\alpha \rightarrow 0$, we obtain the original infrared divergence for $0 \leq \gamma < 1$. In particular, this means that the method outlined in Section 2 is valid either with no regularization or renormalization only for positive pressures. As a matter of fact, although the regularized quantities P_α and ρ_α are finite, they are cutoff dependent. To eliminate this dependence, a renormalization scheme is required. In this connection it seems interesting to investigate the second quantization of γ -fluids and search for a renormalization scheme in this theory. This issue is presently under investigation.

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